Sign-independent synchronization in unidirectionally coupled chaotic systems

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In this paper, a synchronization scheme independent of the sign of the driving signal is developed and analyzed. The corresponding synchronization phenomenon is termed sign-independent synchronization in this context. In our coupling scheme, an even coupling function is used at the response in an attempt to make the response independent of the sign of the driving signal. Furthermore, the Jacobian (and hence the stability) defined on the synchronization manifold of sign-independent synchronization systems can be made identical to that of a corresponding *active-passive decomposition* system. Hence the conditions for achieving stable sign-independent synchronization can be generally established as a trivial case in unidirectionally coupled chaotic systems where the method of *active-passive decomposition* is applicable. Through numerical experiments performed on both continuous and discrete systems, we demonstrate that in-phase synchronization can be maintained even if the phase of the driving signal is randomly inverted. We believe the proposed chaotic synchronization will find important application in digital communications. [S1063-651X(99)09111-4]

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I. INTRODUCTION

In 1990, Pecora and Carroll [1] demonstrated that certain subsystems of chaotic systems could be made to synchronize by linking them with common signals. Ever since then, synchronization of chaotic systems has become one of the most interesting research topics in the study of chaos. Different synchronization phenomena such as identical (in-phase) synchronization [1–9], antiphase synchronization [10, 11], and generalized synchronization [12–14] were reported. The basic idea of these synchronization methods is to seek a subpart (or a subsystem) which possesses negative conditional Lyapunov exponents (CLE's) on a defined synchronization manifold. Thus the subpart is insensitive to initial conditions. Two representative methods reflecting the captioned idea are *active-passive decomposition* (APD) [2,13] and *parameter adjusting* [7].

In recent years, secure communication via synchronized chaos has been intensely studied. Unfortunately, the sensitivity of chaotic synchronization makes chaotic communication schemes [1,2,9] using analog information not robust to external interference. This is because the quality of the recovered information in such schemes is much affected by the synchronization error, which tends to be significant if the transmission channel is very noisy. Digital chaotic communication schemes, associated with *spread spectrum communication* techniques, were proposed [15–18] with claims of robustness. Their idea is based on the fact that chaotic signals have the features of nonperiodicity and short time correlation, which in the frequency domain correspond to a broadband continuous spectrum. Thus, a chaotic signal is naturally a good candidate for use as a spreading signal in spread spectrum communication. In a chaotic spread spectrum system, a chaotic spreading signal is modulated by multiplication with a binary data sequence which is chosen from $\{-1,+1\}$ and updated at a lower rate (which corresponds to a narrower base band in frequency domain). At the receiver, a replica of the chaotic signal is required so that the binary data can be detected using a correlation detection algorithm. In such detection processes, since only the correlation property of the chaotic signal is used, correct recovery of the binary data can still be possible even if the signal-to-noise ratio is very low.

However, one of the key problems to be addressed in chaotic spread spectrum communication systems is the synchronization of the chaotic spreading signal in both the transmitter and the receiver. Attempts have been made to solve this problem under certain assumptions [15,16] or based on particular models [17,18]. In the inchoate works [15,16] on chaotic spread spectrum systems, the synchronization is taken as *a priori* by assuming that both the initial conditions and the initial time of the chaotic system are known. This assumption can cause difficulty in implementation, especially when *random access* communication or a real chaotic sequence is required. (In some applications, the chaotic system is used to generate chaotic data of a certain length; then the set of chaotic data is stored and repeatedly used. The produced signal is a pseudorandom signal that is not really chaotic in that it is periodic.) Theoretically, the autosynchronization property of chaotic systems naturally meets the requirement of random access. However, autosynchronization seems difficult to achieve in the chaotic spread spectrum system. The fundamental difficulty is incurred by the multiplication of the bipolar data signal, which will change the polarity of the driving signal randomly and then spoil the *in-phase* synchronization if conventional techniques of chaotic synchronization are used directly. Our motivation for this study is to develop an autosynchronization scheme that ensures in-phase synchronization even when the phase of the driving signal is randomly inverted.

II. THEORY

Consider a chaotic drive system

$$\dot{\mathbf{x}} = F(\mathbf{x}), \tag{1}$$

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where $F:R^n \rightarrow R^n$. The transmitted signal is a scalar signal given by $h_{\mathbf{x}}=g(\mathbf{x})$ $(g:R^n \rightarrow R)$. Our intention is to construct a response system $\mathbf{y}$ whereby sign-independent synchronization (SIS) can be achieved, i.e., the response system can be synchronized to the drive system no matter whether it is driven by $+h_{\mathbf{x}}$ or $-h_{\mathbf{x}}$. We also demand that the switching of the driving signal between $+h_{\mathbf{x}}$ and $-h_{\mathbf{x}}$ should not cause transients during the synchronization process. To achieve SIS, we consider unidirectionally coupled systems with the following conditions: (a) the synchronization state given by $\mathbf{y}=\mathbf{x}$ is an *invariant* subspace defined on the whole space of the coupled systems, say the synchronization manifold; (b) the driving signal is coupled into the response system through a coupling function of even symmetry; (c) the response possesses negative conditional Lyapunov exponents. The captioned conditions are set, respectively, to make sure that the synchronization manifold is (i) existent, (ii) independent of the sign of the driving signal, and (iii) asymptotically stable. As we have explained, SIS can be established as long as the three conditions (a), (b), and (c) are satisfied. At first glance, the coexistence of conditions (b) and (c) seems questionable or nontrivial. It seems really dubious that there *always* exists a coupling function of even symmetry that simultaneously guarantees a passive response system with negative CLE's. In the following context, we shall demonstrate that, by using our designed coupling, the specified conditions not only can coexist but also can be generally established in unidirectionally coupled systems.

Consider the response system given by

$$\dot{\mathbf{y}}=G(\mathbf{y},c(\mathbf{y},h_{\mathbf{x}})), \tag{2}$$

where $G:R^n \times R \rightarrow R^n$ and $G(\mathbf{x},c)=F(\mathbf{x})$ if and only if $c=0$. $c:R^n \times R \rightarrow R$ is a coupling function given in the following form:

$$c(\mathbf{y},h_{\mathbf{x}})=\frac{v(h_{\mathbf{x}})-v(h_{\mathbf{y}})}{v'(h_{\mathbf{y}})}, \tag{3}$$

where $h_{\mathbf{y}}=g(\mathbf{y})$. Assume that $v:R \rightarrow R$ is a continuous, piecewise differentiable function of even symmetry, i.e., $v(h)=v(-h)$. $v'(h)$ is the derivative of $v(h)$. With such a coupling, synchronization can occur on an invariant subspace given by $\mathbf{y}=\mathbf{x}$. Note that the coupling function is even symmetric with respect to $h_{\mathbf{x}}$, i.e., $c(\mathbf{y},h_{\mathbf{x}})=c(\mathbf{y},-h_{\mathbf{x}})$, hence the state variable $\mathbf{y}$ remains independent of the sign of the driving signal. Thus switching between the in-phase and the antiphase transmission causes no transients during the synchronization process. Therefore condition (a) and condition (b) are satisfied using the coupling function defined in Eq. (3). To testify the stability of the synchronization manifold, we investigate the linearized equation that governs the evolution of the deviation $\mathbf{e}=\mathbf{y}-\mathbf{x}$, which is given by a linear differential equation

$$\dot{\mathbf{e}}=DG(\mathbf{x})\mathbf{e},$$

where $DG(\mathbf{x})$, conditioned on $\mathbf{x}$, is the Jacobian matrix defined by the partial derivative of $G(\mathbf{y},c(\mathbf{y},h_{\mathbf{x}}))$ on $\mathbf{y}$ under the synchronization condition $\mathbf{y}=\mathbf{x}$. The CLE's will emerge from the ergodic average of the Jacobian. It is well known [1,2] that, if all CLE's are negative, Eq. (2) represents a passive system. Then the synchronization is linearly stable and the deviation $\mathbf{e}$ converges to zero asymptotically. In the following, we will demonstrate that negative CLE's can be generally guaranteed by using a decomposition which is associated with an APD system and which has the same Jacobian. To show this, we derive that (see the Appendix)

$$DG(\mathbf{x})=DF(\mathbf{x})-\left.\frac{\partial G(\mathbf{x},c)}{\partial c}\frac{dg(\mathbf{x})}{d\mathbf{x}}\right|_{c=0}, \tag{4}$$

where $DF(\cdot)$ is the Jacobian of $F(\cdot)$. Note that the Jacobian defined by Eq. (4) is independent of $v(\cdot)$ and $v'(\cdot)$. Furthermore, it is not difficult to show that this Jacobian is the same as that in Eq. (5) almost everywhere [we say ''almost everywhere'' because $v(\cdot)$ is piecewise differentiable, but these break points have zero Lebesgue measure],

$$\dot{\mathbf{y}}=G(\mathbf{y},h_{\mathbf{x}}-g(\mathbf{y})), \tag{5}$$

where the coupling function is changed to $c(\mathbf{y},h_{\mathbf{x}})=h_{\mathbf{x}}-g(\mathbf{y})$, while the decomposition $G(\cdot,\cdot)$ remains the same as in Eq. (2). Obviously, the synchronization manifold exists with such a response system and possesses the same CLE's as in Eq. (2), although Eq. (5) does not permit SIS in general. Note that Eq. (5) can always be rewritten as

$$\dot{\mathbf{y}}=G_a(\mathbf{y},h_{\mathbf{x}}), \tag{6}$$

which is a form of APD [2]. The transformation between Eq. (5) and Eq. (6) is equivalent since given either form, the other can then be derived by using

$$G_a(\mathbf{y},h_{\mathbf{x}})=G(\mathbf{y},h_{\mathbf{x}}-g(\mathbf{y}))$$

or

$$G(\mathbf{y},c)=G_a(\mathbf{y},c+g(\mathbf{y})).$$

Thus, for a given APD system in the form of Eq. (6) with known $G(\cdot,\cdot)$ and $g(\cdot)$, a SIS system can then be constructed by using a coupling function of even symmetry in the form of Eq. (3). Particularly, if we set $v(\cdot)$ to be the absolute function, the coupling function will be

$$c(\mathbf{y},h_{\mathbf{x}})=\mathrm{sgn}[g(\mathbf{y})]h_{\mathbf{x}}-g(\mathbf{y}) \tag{7}$$

and the corresponding SIS system

$$\dot{\mathbf{y}}=G_a(\mathbf{y},\mathrm{sgn}[g(\mathbf{y})]|h_{\mathbf{x}}|)$$

is then established.

So far, the SIS scheme has been introduced and analyzed, and the relationship between the SIS scheme and the APD has been studied. It is concluded that SIS can be established in coupled systems where the APD method in the form of Eq. (6) is applicable. The generality of APD in unidirectionally coupled systems has been studied in [2,13], based on which we conclude that SIS can also be generally established in uniderectionally coupled systems. In the following section, we will demonstrate the idea of SIS with numerical experiments.

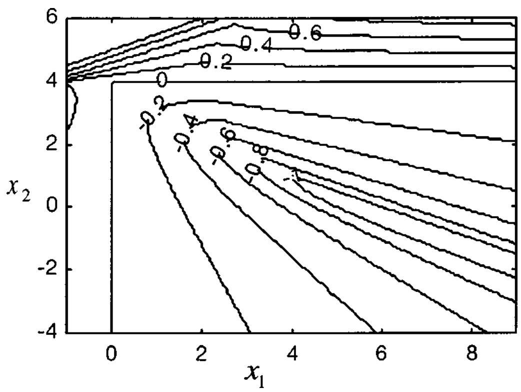

FIG. 1. Distribution of the maximum real part of the eigenvalues of the Jacobian which is conditioned on x_1 and x_2.

III. NUMERICAL EXPERIMENTS

In the following examples, we set $v(h)=|h|$. The coupling function is then written in the form of Eq. (7). Therefore the response system is confirmed, if both $g(\cdot)$ and $G(\cdot,\cdot)$ are given.

Example 1: Rössler system. Consider the Rössler system,

$$\dot{x}_1=x_1(x_2-4)+2,$$
$$\dot{x}_2=-x_1-x_3, \qquad (8a)$$
$$\dot{x}_3=x_2+0.45x_3.$$

Let

$$g(\mathbf{x})=x_3+0.9,$$

where the constant bias 0.9 is introduced to set $h_{\mathbf{x}}$ to zero mean, as required in spread spectrum communication. By applying Eq. (7), the coupling function is

$$c(\mathbf{y},h_{\mathbf{x}})=\mathrm{sgn}(y_3+0.9)|h_{\mathbf{x}}|-y_3-0.9.$$

We set the decomposition to

$$G{:}G(\mathbf{y},c)=F(\mathbf{y})+[0\ 0\ 0.45c]^T.$$

The response for SIS is then given by

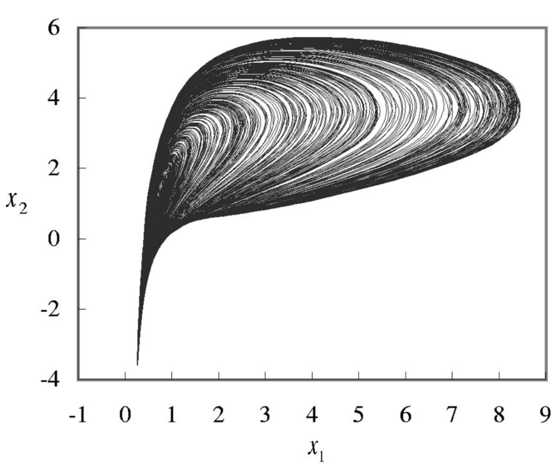

FIG. 2. Projection of Rössler attractor on x_1, x_2 plane.

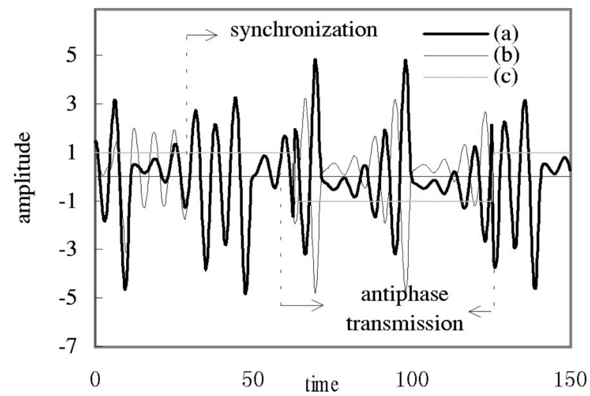

FIG. 3. (a) Modulated signal $b(t)h_{\mathbf{x}}$; (b) recovered chaotic carrier $h_{\mathbf{y}}$; (c) bipolar data $b(t)$.

$$\dot{y}_1=y_1(y_2-4)+2,$$
$$\dot{y}_2=-y_1-y_3, \qquad (8b)$$
$$\dot{y}_3=y_2+0.45\,\mathrm{sgn}(y_3+0.9)|h_{\mathbf{x}}|-0.405,$$

where the Jacobian matrix is

$$DG(\mathbf{x})=\begin{bmatrix} x_2-4 & x_1 & 0\\ -1 & 0 & -1\\ 0 & 1 & 0\end{bmatrix},$$

which is conditioned on x_1 and x_2. The distribution of the maximum real part of the eigenvalues is depicted in Fig. 1. The projection of the Rössler attractor on the x_1-x_2 plane is shown in Fig. 2. On average, the response system has a negative maximum CLE ($\lambda_{\max}=-0.12$), hence synchronization can be realized in practice. The results of our synchronization experiment performed on the coupled Rössler systems are shown in Figs. 3 and 4, where the synchronization error defined by $\|\mathbf{x}-\mathbf{y}\|$ decays to zero at a steady state. The synchronization is maintained during both the in-phase and the antiphase transmission without any transient after each switching. To test the robustness of the synchronization, we add a random noise bounded within -0.15 and $+0.15$ to the driving signal. We do not observe a large synchronization error burst as reported in Ref. [3] once synchronization is

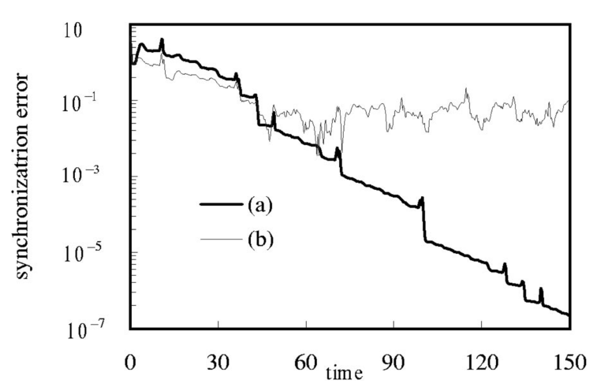

FIG. 4. Experiment on coupled Rössler systems. (a) Synchronization error given by $\|\mathbf{x}-\mathbf{y}\|$ without noise. (b) Synchronization error under weak additive noise bounded within $(-0.15,+0.15)$.

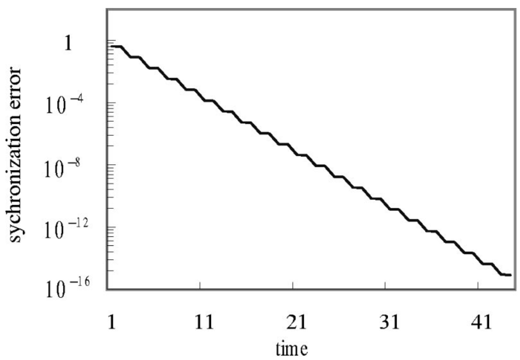

FIG. 5. The SIS experiment performed on the coupled Hénon maps. Synchronization error defined by $\|\mathbf{x}(n)-\mathbf{y}(n)\|$.

attained. The synchronization error is observed to be the same order as the additive noise [curve (b) in Fig. 4].

According to the relationship between SIS and APD discussed in Sec. II, the y_3 component in the corresponding APD system is derived to be

$$\dot{y}_3 = y_2 + 0.45x_3 .$$

Obviously, the corresponding response possesses the same Jacobian matrix as that in Eq. (8b). Synchronization with such a response system was reported as an example of APD in Ref. [2].

Example 2: Henon map. Similarly, SIS can also be established in discrete chaotic systems, for example the Henon map formulated by

$$x_1(n+1) = 0.3x_2(n) - x_1(n)^2 + 1.4,$$

$$x_2(n+1) = x_1(n).$$

We let

$$g(\mathbf{x}) = -x_1^2 + 0.5x_2 + 1$$

and

$$G{:}G(\mathbf{y},c) = F(\mathbf{y}) + [c\ 0]^T.$$

Then, by applying Eq. (7), the response is

$$y_1(n+1) = -0.2y_2(n) + 0.4 + \mathrm{sgn}[-y_1(n)^2 + 0.5y_2(n) + 1]|h_{\mathbf{x}}(n)|,$$

$$y_2(n+1) = y_1(n).$$

The Jacobian of the nonautonomous system is $\left[\begin{smallmatrix} 0 & -0.2 \\ 1 & 0 \end{smallmatrix}\right]$, eigenvalues of which are all confined to within the unit circle ($|\lambda|^2 = 0.2$). Since the Jacobian is time invariant, the Lyapunov multiplier is the same as the modulus of the eigenvalue. Therefore, the response is a *passive* system and any deviation from the synchronization manifold will asymptotically vanish. It is verified in Fig. 5 that the synchronization error, defined by $\|\mathbf{y}(n)-\mathbf{x}(n)\|$, trails off exponentially at a rate of α^n ($\alpha \approx 0.4472$). In our numerical experiments, we find that the convergence of the synchronization error obeys the exponential law for any randomly chosen initial conditions, which means that the synchronization is of the robust kind [3].

Recently, Zhou and Chen [18] proposed a digital chaotic communication system based on in-phase and antiphase synchronized Hénon maps, and successfully demonstrated the robustness of the system to additive noise and quantization error. In their scheme, the recovered chaotic carrier is just correlated but not identical to the original during ‘‘-1’’ transmission even if no noise is present. A transient is inevitably induced after each switch between in-phase transmission and antiphase transmission. In addition, the conditions that allow their idea cannot be generally established in most chaotic models and thus the method is not general. In contrast with their synchronization scheme, SIS can be generally established in unidirectionally coupled systems. The chaotic carrier in either in-phase ($+1$) or antiphase (-1) transmission can be fully recovered, and switching between different transmission modes causes no transient during the synchronization process, which means better performance in potential communication applications. Besides, the synchronization error in our experiment on Hénon maps also converges to zero exponentially. Thus it can be predicted that robustness similar to that of Zhou and Chen can be obtained.

In the two examples, stability of the synchronization is examined by CLE’s. This relies on an ergodic average and is generally obtained numerically. In the following example, we will show that the stability of the SIS can also be proved rigorously using the method of contraction maps [10].

Example 3: Logistic map. The dynamics of zero mean logistic maps are

$$x_{n+1} = 1 - 2x_n^2 .$$

Let

$$g(x) = x.$$

One decomposition for SIS is

$$G(y,c) = 1 - 2y^2 - 4yc.$$

Using Eq. (7), the response is then formulated as

$$y_{n+1} = 1 + 2y_n^2 - 4|x_n y_n|.$$

Note that $|y_{n+1} - x_{n+1}| = 2(|x_n| - |y_n|)^2 \leqslant 2|y_n - x_n|^2$. Obviously, once $|y_n - x_n| < 1/2$, the quantity $|y_n - x_n|$, controlled by a contraction map, converges to zero. From the ergodic property of chaotic systems, this condition can always be satisfied for randomly chosen initial conditions supposing the response is bounded. Thus SIS can be achieved.

IV. DISCUSSION

Note that, if $g(\cdot)$ and $F(\cdot)$ are odd symmetric, i.e., $g(\mathbf{x}) = -g(-\mathbf{x})$ and $F(\mathbf{x}) = -F(-\mathbf{x})$, since $c(\pm\mathbf{x}, h_{\mathbf{x}}) = 0$, and $\pm\mathbf{x}$ are both trajectories of the system, two invariant subspaces will emerge defined, respectively, by $\mathbf{y} = \mathbf{x}$ and $\mathbf{y} = -\mathbf{x}$, the in-phase synchronization manifold and antiphase synchronization manifold. Owing to the symmetric property of the system, the coupled systems will have identical stability on the two manifolds. In the event that both manifolds are stable, the eventual state of the coupled systems will be of either in-phase synchronization or antiphase synchronization, depending only on the initiation point of the trajectory. In such systems, the boundary between the attracting basins of the two synchronization manifolds may have complicated fractal structure [19]. Thus any disturbance may cause the trajectory to switch randomly between the two basins. This may cause error in potential communication applications. However, in most cases the antiphase synchronization can be easily removed from the coupled system by selecting a function $g(\cdot)$ that is not odd symmetric. For example, the addition of a constant bias to $g(\mathbf{x})$ generally causes a loss of symmetry so that the antiphase synchronization, $\mathbf{y}=-\mathbf{x}$, is no longer an invariant subspace of the coupled systems, because a trajectory initiated from the subspace cannot continue to remain on it.

Robustness of chaotic synchronization always varies for different choices of model or different forms of coupling. It can be predicted that better robustness can be obtained if the synchronization is globally stable having negative CLE's with as large a magnitude as possible. The choice of function $v(\cdot)$ also influences the robustness of the synchronization, although the Jacobian of the response is independent of $v(\cdot)$ in the proposed SIS system. The reason is that the Jacobian only describes the variation of a deviation that is assumed infinitesimal. When the deviation grows larger, $v(\cdot)$ will affect the synchronization. For simplicity and ease of implementation by circuitry, we set $v(\cdot)$ to be the absolute function. However, if we choose $v(h)$ to be differentiable at $h=0$, due to the symmetric property of the function, $v'(0)$ will be equal to 0, which will cause the control signal produced by the coupling function to be infinite. This can be avoided by turning off the control and letting the response run autonomously in the region about $h_y=0$. Synchronization can still be maintained provided the area of the region is chosen appropriately to ensure negative CLE's.

We have observed in numerical simulation that the communication system based on sign-independent synchronized Hénon maps is not very sensitive to parameter mismatch. This reminds us of another issue commonly concerned with chaotic communication, that is, the security. It is required that the receiver system be very sensitive to parameter mismatch, so that a third party can lock into the communication with a probability approaching zero. (We assume that the eavesdropper has known in advance the structure of the system and means to lock into the communication by trying every parameter in some defined region in the parameter space). On the other hand, robustness requires that the receiver system be insensitive to parameter mismatch and external interference, otherwise the communication will be very difficult to establish in practice. The two aspects seem incompatible in this sense. Thus, as a trade off, such a communication system can still be considered secure by assuming that the third party does not know the structure of the system, or that the eavesdropper will be frustrated if he tries to retrieve the data using conventional linear signal processing methods. Moreover, our design is based on active-passive decomposition, which allows the flexibility to use high-dimensional hyperchaotic systems and will then make it more difficult for a third party to detect the structure of the system [8,9]. In the communication applications in which continuous chaotic signals were used, the bipolar data signal can be smoothed out before modulation to avoid the sharp transition of the transmitted signal between in-phase and antiphase transmissions that would be easy to detect. Such manipulation will cause transients, but will not affect the correct recovery of the data as long as the duration of one data bit is set large enough compared with the transient time.

It should be pointed out that sign-independent synchronization is different from the antiphase synchronization reported in [10,11,18]. In antiphase synchronization, the state variables of the two chaotic systems, namely drive and response, have the same amplitude but opposite sign. In sign-independent synchronization, however, *in-phase* synchronization is established even if the phase of the driving signal is randomly inverted.

In this paper, an autosynchronization scheme has been established aiming to solve the synchronization problem in chaotic spread spectrum communication systems where the driving signal is modulated by a bipolar data sequence. Verification is conducted through numerical experiments performed on several commonly used chaotic models. By deriving an identical Jacobian in a corresponding APD system, we conclude that sign-independent systems can be generally established as a trivial case in unidirectionally coupled chaotic systems. Our research is ongoing, and we shall explore its potential applications in the near future.

ACKNOWLEDGMENT

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APPENDIX

$$DG(\mathbf{x})=\left.\frac{\partial G(\mathbf{y},c(\mathbf{y},h))}{\partial \mathbf{y}}\right|_{\mathbf{y}=\mathbf{x}}$$

$$=\left.\frac{\partial G(\mathbf{y},c)}{\partial \mathbf{y}}+\frac{\partial G(\mathbf{y},c)}{\partial c}\frac{\partial c(\mathbf{y},h)}{\partial \mathbf{y}}\right|_{\mathbf{y}=\mathbf{x}},$$

where $h=b^k g(\mathbf{x})$. Since $c(\mathbf{y},h)=[v(h)-v(g(\mathbf{y}))]/v'(g(\mathbf{y}))$, then

$$\frac{\partial c(\mathbf{y},h)}{\partial \mathbf{y}}=-\frac{dv(g)}{dg}\frac{dg(\mathbf{y})}{d\mathbf{y}}\frac{1}{v'(g)}$$

$$+[v(h)-v(g(\mathbf{y}))]\frac{\partial}{\partial \mathbf{y}}\left(\frac{1}{v'(g(\mathbf{y}))}\right).$$

Under the synchronization condition $\mathbf{y}=\mathbf{x}$, we have

$$v(h)-v(g(\mathbf{y}))=0,$$

$$c(\mathbf{y},h)=0,$$

and note

$$v'(g)\equiv\frac{dv(g)}{dg}.$$

Then

$$\frac{\partial c(\mathbf{y},h)}{\partial \mathbf{y}} = -\frac{dg(\mathbf{y})}{d\mathbf{y}}.$$

Recall that $G(\mathbf{x},0)=F(\mathbf{x})$. Then we get

$$DG(\mathbf{x}) = \frac{dF(\mathbf{x})}{d\mathbf{x}} - \frac{\partial G(\mathbf{x},c)}{\partial c}\frac{dg(\mathbf{x})}{d\mathbf{x}}\bigg|_{c=0}.$$

[1] L. M. Pecora and T. L. Caroll, Phys. Rev. Lett. **64**, 821 (1990); Phys. Rev. A **44**, 2374 (1991); T. L. Caroll and L. M. Pecora, Physica D **67**, 126 (1993).

[2] L. Kocarev and U. Parlitz, Phys. Rev. Lett. **74**, 5028 (1995); L. Kocarev, U. Parlitz, and T. Stojanovski, Phys. Lett. A **217**, 280 (1996); U. Parlitz, L. Kocarev, T. Stojanovski, and H. Preckel, Phys. Rev. E **53**, 4351 (1996).

[3] M. M. Sushchik, Jr., N. F. Rulkov, and H. D. I. Abarbanel, IEEE Trans. CAS **44**, 867 (1997).

[4] M. K. Ali and J. Q. Fangm, Phys. Rev. E **55**, 5285 (1997).

[5] R. Brown and N. F. Rulkov, Phys. Rev. Lett. **78**, 4189 (1997).

[6] R. He and P. G. Vaidya, Phys. Rev. A **46**, 7387 (1992).

[7] Z. Liu and S. Chen, Phys. Rev. E **55**, 6651 (1997).

[8] J. H. Peng, E. J. Ding, M. Ding, and W. Yang, Phys. Rev. Lett. **76**, 904 (1996).

[9] L. M. Pecora, T. L. Carroll, G. Johnson, and D. Mar, Phys. Rev. E **56**, 5090 (1997).

[10] T. Ushio, Phys. Lett. A **198**, 14 (1995).

[11] L. Y. Cao and Y. C. Lai, Phys. Rev. E **58**, 382 (1998).

[12] N. F. Rulkov, M. M. Shushchik, L. S. Tsimring, and H. D. I. Abarbanel, Phys. Rev. E **51**, 980 (1995).

[13] L. Kocarev and U. Parlitz, Phys. Rev. Lett. **76**, 1816 (1996).

[14] B. R. Hunt, E. Ott, and J. A. Yorke, Phys. Rev. E **55**, 4029 (1997).

[15] G. Heidari-Bateni and C. D. Mcgillem, IEEE Trans. Commun. **43**, 1524 (1994).

[16] U. Parlitz and S. Engeziner, Phys. Lett. A **188**, 146 (1994).

[17] V. Milanovic, K. M. Syed, and M. E. Zaghloul, Int. J. Bifurcation Chaos Appl. Sci. Eng. **7**, 215 (1997).

[18] C. S. Zhou and T. L. Chen, Phys. Lett. A **225**, 60 (1997); Phys. Rev. E **56**, 1599 (1997).

[19] S. W. McDonald, C. Grebogi, E. Ott, and J. A. Yorke, Physica D **17**, 125 (1985).